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UNCOUNTABLE CONSTRUCTIONS FOR B.A. e.c. GROUPS AND BANACH SPACES

BY
SAHARON SHELAH†
Department of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel; Department of Mathematics, University of Wisconsin, Madison, Wisconsin, USA; and Department of Mathematics, University of California, Berkeley, California, USA

ABSTRACT

This paper has two aims: to aid a non-logician to construct uncountable examples by reducing the problems to finitary problems, and also to present some construction solving open problems. We assume the diamond for $\aleph_1$ and solve problems in Boolean algebras, existentially closed groups and Banach spaces. In particular, we show that for a given countable e.c. group M there is no uncountable group embeddable in every G $L_{\infty,\omega}$-equivalent to M; and that there is a non-separable Banach space with no $\aleph_1$ elements, no one being the closure of the convex hull of the others. Both had been well-known questions. We also deal generally with inevitable models (§4).

§1. The general principle

DEFINITION 1.1. (1) We call K a *nice family* if:

(a) K is a family of f.g. models of fixed language (signature).

(b) K is closed under isomorphism (i.e. $M \cong N$, $N \in K$ implies $M \in K$) and taking f.g. submodels.

(c) Each $M \in K$ is countable and the signature is countable and K has, up to isomorphism, $\aleph_0$ members.

(2) K is a very nice family if in addition:

(d) K has the joint embedding property (i.e., if $M_0, M_1 \in K$, then for some $N \in K$ there are embeddings $f_l : M_l \to N$, $l = 0,1$).

(e) K has the amalgamation property (i.e., if $M \subseteq M_l$ ($l = 1,2$) then there is N, $M \subseteq N$ and embeddings f_l of M_l into N over M, i.e. $f_l \restriction M =$ the identity).

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REMARK. If K is nice and has the amalgamation property, then it is the disjoint union of very nice classes.

DEFINITION 1.2. (1) R is called K-nice if

(a) K is a nice family.

(b) R is a set of triples of the form (M, N, a), such that $M \subseteq N$ are in K, and $a \in N - M$. We write $(M, N, a) \in R$ or $R(M, N, a)$ holds.

(c) If $M \subseteq M' \in K$ and $(M, N, a) \in R$ *then* there are N', f such that $M' \subseteq N'$ and f is an embedding of N into N' over M, $f(a) \notin M'$, and $R(M', N', f(a))$ holds.

(d) R is not empty.

(2) R is K-very nice if in addition $R(M, N, a)$, $N \subseteq N'$ implies $R(M, N', a)$.

REMARK. The intuitive meaning of $R(M, N, a)$ is that in some uncountable M^*, $M \subseteq M^*$, there are uncountably many $a' \in M^*$ "similar" to a over M.

DEFINITION 1.3. For a nice family K let

$$K^1 = \{M : \text{every f.g. submodel of } M \text{ is in } K\}$$

(equivalently, $M \in K'$ is a direct limit of members of K),

$$K^H = \{M \in K^1 : \text{if } M_0 \subseteq N_0,\ N_0 \in K,\ M_0 \subseteq M, \text{ and } M_0 \in K \text{ then we can embed } N_0 \text{ into } M \text{ over } M_0\},$$

$$K^x(\lambda) = K^x_\lambda = \{M \in K^x : \|M\| = \lambda\} \qquad \text{for } x = 1, H.$$

DEFINITION 1.4. For $M \subseteq N$ in K^1 $R(M, N, a)$ means: for every f.g. $N_0 \subseteq N$ there are f.g. N_1, M_1 such that $N_0 \subseteq N_1 \subseteq N$, $N_0 \cap M \subseteq M_1 \subseteq N_1 \cap M$, and $R(M_1, N_1, a)$ holds.

DEFINITION 1.5. (1) A *candidate* $\bar{M}$ is a sequence of the form $(M_k, \dots, M_0, a_{k-1}, \dots, a_0)$ such that $R(M_l, M_{l+1}, a_l)$ holds for $l < k$ (hence $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k$, $a_l \in M_{l+1} - M_l$) and each M_l in K. So $\bar{M}^i$ will be $(M^i_k, \dots, M^i_0, a^i_k, \dots, a^i_0)$ etc. We denote k by $k(\bar{M})$, and M_k (or $|M_k|$) by $\operatorname{dom}(\bar{M})$.

(2) If $\bar{M}, \bar{M}'$ are candidates then $\bar{M} \leqq \bar{M}'$ means $k(\bar{M}) = k(\bar{M}')$ and $a_l = a'_l$ (for $l < k(M)$) and $M_l \subseteq M'_l$ (for $l \leqq k$). Also $\bar{M} \leqq^* \bar{M}'$ means there is a one-to-one monotone function h from $\{0, \dots, k(\bar{M}), \dots\}$ into $\{0, \dots, k(\bar{M}'), \dots\}$, such that $a_l = a'_{h(l)}$, $M_l \subseteq M'_{h(l)}$. Clearly those are partial orders and $\bar{M} \leqq \bar{M}'$ implies $\bar{M} \leqq^* \bar{M}'$.

(3) We call $\bar{M}$ an M^*-candidate if $M_0 \subseteq M^*$. "An embedding of (from) $\bar{M}$" means "an embedding from M_k", $\bar{b} \in \bar{M}$ means $\bar{b} \in M_k$, etc.

(4) We call $\bar{M}, \bar{M}'$ *equivalent* if $k(\bar{M}) = k(\bar{M}')$, $M_l = M'_l$ for $l \leqq k$ and for every $M''_k, \dots, M''_0$ the following are equivalent:

(i) $\bar{M} \leqq (M''_k, \dots, M''_0, a_{k-1}, \dots, a_0)$,

(ii) $\bar{M}' \leqq (M''_k, \dots, M''_0, a'_{k-1}, \dots, a'_0)$.

Clearly this is an equivalence relation.

(5) Let $M \leqq^w \bar{M}''$ if for some $\bar{M}'$ equivalent to $\bar{M}''$, $\bar{M} \leqq^* \bar{M}'$. Clearly $\leqq^w$ is a partial order, and $\bar{M} \leqq^w \bar{M}' \Leftarrow \bar{M} \leqq^* \bar{M}'$.

(6) We call f an embedding of $\bar{M}^0$ into $\bar{M}^1$ if for some strictly increasing h: $\{0, \dots, k(\bar{M}^0)\} \to \{0, \dots, k(\bar{M}^1)\}$, f embeds M^0_l into $M^1_{h(l)}$, $f(a^0_l) = a^1_{h(l)}$.

Definition 1.6. Let $M^* \in K^1$, I a set of m-sequences from M^*. We call $(\bar{M}, \bar{b})$ a *witness* for (M^*, I) if $\bar{M}$ is an M^*-candidate, $\bar{b}$ an m-sequence from $\bar{M}, \bar{b} \not\in M_0$ and $\bar{M} \leqq \bar{M}'$, $\bar{M}'$ an M^*-candidate implies there is an embedding f of $\bar{M}'$ into M^* over M'_0, such that $f(\bar{b}) \in I$, and $\bar{b} \not\in M'_0$.[†]

Main Theorem 1.7. *Assume $\diamondsuit_{\aleph_1}$ holds. Suppose R is K-nice. Then there is a model M^* such that*:

(a) $M^* \in K^1(\aleph_1)$.

(b) *For any $m < \omega$ and uncountable set I of m-sequences from M^*, there is a witness for (M^*, I).*

(c) $M^* = \bigcup_{\alpha<\omega_1} M_\alpha$, M_α $(\alpha < \omega_1)$ *increasing and continuous, each M_α is countable, $M_\alpha \in K^1(\aleph_0)$ and for some b_α, $R(M_\alpha, M_{\alpha+1}, b_\alpha)$ holds.*

(d) *If $M \subseteq M_\alpha$, $R(M, N, a)$ then there is an embedding f of N into M^* over M, such that for some β, $f(a) = b_\beta$.*

(e) *If K is very nice and R is K-very nice then each M_α is in K^H.*

(f) $M_0 = \bigcup_{n<\omega} M^n_0$ *and if $M^n_0 \subseteq N \in K$, N cannot be embedded into M_0 over M^n_0 then for some $k > n$, M^k_0, N cannot be amalgamated over M^n_0, $N \in K$ cannot be embedded into M_0 if for some k, M^k_0, N cannot be amalgamated.*

Proof. See last section.

When applying the theorem, it is helpful to notice the following two claims:

Claim 1.8. (1) *If $(\bar{M}, \bar{b})$ is a witness for (M^*, I) then so is every $(\bar{M}', \bar{b})$ where $\bar{M} \leqq \bar{M}'$, $\bar{M}'$ an M^*-candidate. Moreover if $\bar{M} \leqq \bar{M}'$ then there is an M^*-candidate $\bar{M}''$ isomorphic to $\bar{M}'$ over $\bar{M}$, provided that K is very nice.*

(2) *So applying* (c) *of* 1.2(1), *if we are given some finite $A \subseteq M^*$ we can assume our witness $(\bar{M}, \bar{b})$ is such that $A \subseteq M_0$.*

[†]Note that the last "$\bar{b} \not\in M'_0$" says, in particular, that $\bar{b}$ is not in M_0 in a quite strong sense. In fact if K is very nice it says: "if $\bar{M} \leqq \bar{M}'$ then $\bar{b} \not\in M'_0$" (we could also have this taken as a definition).

(3) *In* (1), $\bar{M} \leqq^* \bar{M}'$ *and even* $\bar{M} \leqq^w \bar{M}'$ *suffices.*

CLAIM 1.9. *Suppose* $(\bar{M}, \bar{b})$ *is a witness for* (M^*, I), $k = k(\bar{M})$ *and* $N \in K^1$ *and there are embeddings* f_n *of* $\bar{M}$ *into* N *over* M_0 *for* $n < \alpha$, *where* $\alpha \leqq \omega$ *such that:*

(∗) *for* $l < k$, $n < \alpha$,

$$R\left(\left\langle \bigcup_{m<n} f_m(M_k) \cup f_n(M_l)\right\rangle_N, \left\langle \bigcup_{m<n} f_m(M_k) \cup f_n(M_{l+1})\right\rangle_N, f_n(a_l)\right)$$

or even

(∗)′ *there are* $N_i \subseteq N$ *increasing,* $f_n(M_l) \subseteq N_{k(n-1)+l}$ *such that*

$$R(N_{k(n-1)+l}, N_{k(n-1)+l+1}, f_n(a_l)) \quad \text{holds} \qquad (\text{for } n < \alpha,\ l \leqq k).$$

Then *there is an embedding* g *of* $\langle \bigcup_n f_n(M_k)\rangle_N$ *into* M^* *over* M_i *such that* $g(f_n(\bar{b})) \in I$ *for every* n.

We shall call $(f_n(\bar{M}), f_n(\bar{b}))$ *copies of* $(\bar{M}, \bar{b})$ *over* M_0.

DEFINITION 1.10. R is trivial (for K) if $R = \{(M, N, a) : M \subseteq N,\ a \in N - M,\ M \in K,\ N \in K\}$.

CLAIM 1.11. *Suppose* K *is nice.*

(1) *There is a unique trivial* R, $R_T(K)$.

(2) $R_T(K)$ *is* K*-nice iff* 1.2(c) *holds and* K *has a non-maximal element.* (*Note* 1.2(c) *is a slight strengthening of the amalgamation property for* K.) $R_T(K)$ *is* K*-very nice iff* $R_T(K)$ *is nice.*

(3) $\bar{M}, \bar{M}'$ *are equivalent* (*for* $K, R_T(K)$), *iff* $k(\bar{M}) = k(\bar{M}')$, $M_l = M'_l$ (*for* $l \leqq k$), *and* $a_l \in \langle M_l, a'_l\rangle$, $a'_l \in \langle M_l, a_l\rangle$ (*for* $l < k$).

CONCLUDING REMARKS. (1) We could have worked in a more general context (e.g., the objects will be in a category of models, the morphisms may be homorphisms), or a less general one (e.g., deal with very nice K and R).

(2) When the author lectured this in Madison 79, he took a somewhat different approach, natural for model theory rather than for application to algebra. The theorem says that for any countable (and consistent) $T \subseteq L(Q)$, there is an uncountable model M^* s.t.:

(∗) for any $m < \omega$ and uncountable set I of m-sequences from M, there is a witness $\varphi(x_0, \bar{y}, \ldots, \bar{z}, \bar{b})$ ($\bar{b} \in M$) which means: for any countable $A \subseteq M$,

$$M \models (Qx_0)(\exists \bar{y}_0)(Qx_1)(\exists \bar{y}_1) \cdots (\exists \bar{z} \notin A)\varphi(x_0, \bar{y}_0, \ldots, x_n, y_n, \bar{z}, \bar{b})$$

and if $\bar{c} \in M$, and for every countable $A \subseteq M$,

$$M \models Qx_0 \exists \bar{y}_0 \cdots (\exists \bar{z} \notin A)[\varphi(x_0, \bar{y}_0, \ldots, \bar{z}, \bar{b}) \wedge \psi(x_0, y_0, \ldots, \bar{z}, \bar{c})]$$

then for any countable $A \subseteq M$

$$M \models Qx_0 \exists \bar{y}_0 \cdots Qx_n \exists \bar{y}_n (\exists \bar{z} \in I)[\bar{z} \notin A \wedge \varphi(x_0, \bar{y}_0, \ldots, \bar{z}, b) \wedge \varphi(x_0, \bar{y}_0, \ldots, \bar{z}, \bar{c})].$$

(3) By small changes in the proof of 1.7 we can guarantee that in Definition 1.6 we can demand $\bar{b} \notin A$ for any countable $A \subseteq M^*$.

§2. Boolean algebras: the old material

Baumgartner and Komjath [1] proved

THEOREM 2.1. *Assume $\diamond(\aleph_1)$. There is an uncountable Boolean algebra B, such that it has no uncountable chain or antichain, i.e., among any $\aleph_1$ elements there are two comparable and two incomparable (in the natural partial order of B).*

Rubin [7] got stronger results, e.g.,

THEOREM 2.2. *Assume $\diamond(\aleph_1)$. There is an uncountable Boolean algebra B, such that among any $\aleph_1$ elements of B there are three distinct ones a, b, c such that $a \cap b = c$.*

DISCUSSION. How does Theorem 1.7 help to prove such theorems?

By 1.7, 1.9 any problem of the form "find an uncountable model M^* such that among any $\aleph_1$ elements there are n such that..." can be reduced to a problem of amalgamation of candidates (which are finitely generated). We let K be the family of all finite Boolean algebras and R be trivial, i.e.,

$$R = \{(M, N, a): M, N \in K, M \subseteq N, a \in N - M\}.$$

Obviously (check the definitions):

CLAIM 2.3. *K is very nice and R is K-very nice.*

Our task is very clear. We are given a candidate $\bar{M}$ and $b \in \bar{M}$.[†] For Theorem 2.1 we need to amalgamate two copies of $(\bar{M}, b)$, $(\bar{M}^l, b^l)$ $(l = 1,2)$, over M_0 so that $b^1 \leqq b^2$ and so that (in another amalgamation) b^1, b^2 are incomparable. However the amalgamation has to satisfy $(*)$ from 1.9, which means, in our context, that $a_l^2 \notin \langle M_k^1, M_l^2 \rangle$. As the class of Boolean algebras is a variety, if there is such an amalgamation it is the free amalgamation of M_k^1, M_k^2 over M_0 $(k = k(\bar{M}))$ with the added relevant relation ($b^1 \leqq b^2$ in the first case, none in the second).

[†]Of course, such that $\bar{M} \leqq \bar{M}' \Rightarrow b \notin M_0'$.

The above discussion is quite general, and we have not used the specifications of our problem.

CLAIM 2.4. *Let K, R be as above. For any $\bar{M}$, there is $\bar{M}'$, $\bar{M} \leqq^{w} \bar{M}'$, and $c_l \in M'_0$ ($l < k =^{\text{def}} (k(\bar{M}))$ such that $k(\bar{M}) = k(\bar{M}')$ and:*

(a) *$c_0, \ldots, c_{k-1}$ are disjoint atoms of M'_0,*

(b) *$M'_l = \langle M'_0, a'_0, \ldots, a'_{l-1} \rangle$,*

(c) *$0 < a'_l < c_l$.*

PROOF. *Case I.* $k = 1$

Remember that any finite Boolean algebra is atomic. So M_0, M_1 are atomic. Choose an atom c of M_0 such that $c \cap a_0 \notin M_0$ (there is such c as otherwise

$$a_0 = \bigcup \{a_0 \cap c : c \in M_0, c \text{ an atom}\} \in M_0,$$

contradiction). Choose atoms c^0, c^1 of M_1, $c^0 \leqq c \cap a_0$, $c^1 \leqq c - a_0$ (exists — as $c \cap a_0 \notin M$ necessarily it is $\neq 0$, c and as M^1 is atomic).

Let $M'_1 = M_1$ and M'_0 be the subalgebra of M_1 generated by $\{c^1 \cup c^2\} \cup \{c' : c'$ an atom of M_1 but $c' \neq c^1, c^2\}$. Now let $c_0 = c^1 \cup c^2$, $a'_0 = c^1$ and we finish.

Case II. Any k.

Let N be a Boolean algebra with exactly k atoms $c^*_0, \ldots, c^*_{k-1}$. Let M^*_l be the free product of M_l and N. So clearly $\bar{M} \leqq \bar{M}^*$ and $c^*_l \cap a_l - \bigcup_{m<l} c^*_m \notin M^*_l$. Now "below" each c^*_l we repeat the previous argument. For each l choose an atom c^0_l of M^*_l such that $c^0_l \leqq c^*_l - \bigcup_{m<l} c^*_m$, $c^0_l \cap a_l \notin M^*_l$. Then choose atoms c^1_l, c^2_l of M^*_k such that $c^1_l \leqq c^0_l$, $c^2_l \leqq c^0_l$, $c^1_l \leqq c^0_l \cap a_l$, $c^2_l \leqq c^0_l - a_l$.

Let $c_l = c^1_l \cup c^2_l$ for $l < k$.

Let M'_m be the subalgebra of M^*_k generated by $\{c : c$ an atom of M^*_k but $c \neq c^1_l, c^2_l$ for $m \leqq l < k\}$. It is easy to check that $\bar{M}' = \langle M^*_h, \ldots, M^*_0, c^1_{k-1}, \ldots, c^1_0 \rangle$ and $c_0, \ldots, c_{k-1}$ satisfy the conclusion of the claim.

Let us return to the theorems.

PROOF OF THEOREM 2.1. Follows by 2.2 because if $a \cap b = c$, a, b, c distinct then $c \leqq a$, and a, b are incomparable.

PROOF OF THEOREM 2.2. We use 1.7 to get M^*, for K, R as above. Let $I \subseteq M^*$ be uncountable so by 1.7(b) there is a witness $(\bar{M}, b)$ for (M^*, I). By 1.8(3), 1.11(3) we can assume $\bar{M}$ is as in 2.4, and for every $l < k$, $a_l \leqq c_l$, and $b \cap c_l \in \{0, a_l, c_l\}$.

Let $\bar{M}^1, \bar{M}^2$ be two copies of $\bar{M}$ freely amalgamated over M_0 and define $\bar{M}^0$ by

$$a^0_l = a^1_l \cap a^2_l, \qquad M^0_l = \langle M^1_0, a^0_0, \ldots, a^0_{l-1} \rangle.$$

It is easy to check that $b^1 \cap b^2 = b^0$ (b^l the copy of b in $\bar{M}^l$), and that (*) of 1.9 holds, so by 1.9 in I there are distinct b^0, b^1, b^2 such that $b^1 \cap b^2 = b^0$, as required.

In fact the above proof proves more:

DEFINITION 2.5. A Boolean algebra B is 1-Rubin if $\|B\| = \aleph_1$, and for every uncountable $I \subseteq B$, there are disjoint non-zero $c_0, \ldots, c_{k-1} \in B$ and $c_k \in B$ disjoint to them so that, for every $c_l^0 < c_l^1 < c_l$ $(l < k)$, there is $x \in I$ such that:

$$\bigcup_{l<k} c_l^0 \cup c_k \leqq x \leqq \bigcup_{l<k} c_k^1 \cup c_k \qquad (\text{so } c_l^0 < x \cap c_l \leqq c_l^1).$$

DEFINITION 2.6. A Boolean algebra B is called m-Rubin if: $\|B\| = \aleph_1$ and for any uncountable set of m-sequences from M there are disjoint $c_0, \ldots, c_{k-1}$ and c_k^n $(n < m)$ disjoint to them and terms

$$\tau_n = \tau_n(x_0, \ldots, x_{k-1}) = \bigcup_{l<k} \tau_n^l$$

for $n < m$ such that:

(a) $\tau_n^l \in \{0, x, c_l - x, c_l\}$,

(b) for every $c_l^1 < c_l^2 < c_l$ $(l < k)$ there are $d_l \in B$, $c_l^1 < d_l < c_l^2$ and $b = (b_0, \ldots, b_{m-1}) \in I$ such that, for $n < m$,

$$b_n = \tau(d_0, \ldots, d_{k-1}) \cup c_k^n.$$

DEFINITION 2.7. A Boolean algebra B is Rubin if it is m-Rubin for every m.

Rubin [7] constructs 1-Rubin Boolean algebras and investigates their properties; the author noted the generalization to n-Rubin (see [10]).

THEOREM 2.8. *Assume $\diamondsuit(\aleph_1)$. Then there is a Rubin Boolean algebra.*

§3. Boolean algebras with extra predicates

When we want to construct Boolean algebras of power $\aleph_1$ with extra properties it is sometimes advisable to expand the Boolean algebras by more relations (or functions).

Baumgartner and Komjath proved in [1] (and Rubin [7] strengthened the result to the best possible one):

THEOREM 3.1. *Assume $\diamondsuit(\aleph_1)$. There is a Boolean algebra B such that:*

(a) $\|B\| = \aleph_1$,

(b) *among any $\aleph_1$ elements of B, there are two comparable and two incomparable elements,*

(c) *$I = \{a : \{c : c \in B, c \leqq a\}$ is countable$\}$ is a maximal ideal.*

PROOF. Let

$$K = \{(B,P): B \text{ a finite Boolean algebra, } P \text{ a maximal ideal of } B\},$$
$$R = \{(M,N,a): M \subseteq N,\ a \in N - M \text{ and } b \in P^M \text{ implies } a \cap b \in M\}.$$

Notice that if $M \in K$, there is a unique atom $c \in M$ such that $P^M = \{b \in M : c \cap b = 0\}$, and if $(M,N,a) \in R$, $a - c \in M$, $a \cap c \notin M$, under this context.

CLAIM 3.2. (1) *K is very nice, R is K-very nice.*

(2) *For any $\bar{M}$ there is $\bar{M}'$, $\bar{M} \leqq \bar{M}'$ and for every $l < k(\bar{M}')$, $M'_{l+1} = \langle M'_l, a'_l \rangle$ s.t.:*

Let c_l be the unique atom in $M'_l - P(M'_l)$. Then $a'_l - c_l \in M'_l$ and M'_l is generated by $M'_0, c_0, \ldots, c_l$, and $c_0 \geqq c_1 \geqq \cdots$, so using 1.8(3) we will be able to assume $a'_l = c_{l+1} - c_l$.

PROOF. (1) Trivial.

(2) Among all $\bar{N}$, $\bar{M} \leqq \bar{N}$, $M_{k(\bar{M})} = N_{k(\bar{M}^*)}$ choose one $\bar{M}^*$ with the maximal M^*_{k-1} and then maximal M^*_{k-2} etc. (exists as $M^{k-2}_{k(\bar{M})}$ is finite). Suppose $M^*_{l+1} \neq \langle M^*_l, a^*_l \rangle$. Let c_l be the unique atom in $M^*_l - P(M^*_l)$, so $1 - c_l \in P$, so $a^*_l - c_l \in M^*_l$. If there is $x \in M^*_{l+1} - M^*_l$, $x \cap c_l = 0$, then we can replace M^*_l by $\langle M^*_l, x \rangle$ (this is allowed — check the definition of R) and we get a contradiction to the choice of $\bar{M}^*$. If not, let $d_0, \ldots, d_m$ be the atoms of M^*_{l+1}, which are $\leqq c_l$, and w.l.o.g. $d_0 \in M^*_{l+1} - P(M^*_{l+1})$. If $m = 0$, $a^*_l \in M^*_l$, contradiction: if $m = 1$ then $M^*_{l+1} = \langle M^*_l, a^*_l \rangle$, contradicting an assumption. So $m > 1$, w.l.o.g. $a_l \cap d_0 \neq a_l \cap d_1$ and define: M'_n is M^*_n if $n \leqq l$, $\langle M^*_l, d_m \rangle$ if $n = l+1$, M^*_n if $n > l+1$ and a'_n is a^*_n if $n < l$, d_1 if $n = l$, a^*_{n-1} if $n > l$; $\bar{M}'$ contradicts the choice of $\bar{M}^*$.

CONTINUATION OF THE PROOF OF 3.1. Totally parallel to that of 2.1, 2.2: from any $\aleph_1$ member elements of $P(M^*)$ there are distinct a, b, c, $a \cap b = c$.

Quite naturally Rubin has asked whether the Baumgartner–Komjath Theorem 2.1 implies his.

THEOREM 3.3. *Assume $\diamondsuit(\aleph_1)$. There is an uncountable Boolean algebra, which is not 1-Rubin, but among any $\aleph_1$ elements there are two comparable elements (in fact there are a, b, c such that $a \cap b = c$).*

REMARK. In fact all conclusions of Rubin [7] on configurations for 1-Rubin (and the parallels for n-Rubin) hold.

PROOF. Let T be the set of the following axioms (P, R one-place, two-place predicates):

The axioms of Boolean algebras

$$R(x,y) \rightarrow x \leqq y$$

$$R(x,y) \wedge x \leqq x_1 \leqq y_1 \leqq x \rightarrow R(x_1,y_1)$$

$$R(x,x) \rightarrow \neg P(x)$$

$$\neg R(0,1)$$

The intended meaning is that P will be an uncountable set exemplifying the Boolean algebras not 1-Rubin, and R a set of intervals disjoint to P.

$$K = \{M : M \text{ a finite model of } T\}.$$

$$R = \{(M,N,a) : M \subseteq N, M \in K, N \in K, a \in N - M\} \qquad \text{(i.e. } \bar{R} \text{ is trivial)}.$$

CLAIM 3.4. (1) *K is very nice and R is K-very nice. In fact if $M \subseteq N_0, N_1$, are in K, the free product M' of N_0, N_1 over M is in K (i.e., as a Boolean algebra it is the free product, $P(M') = P(N_0) \cup P(N_1)$ and $R(M') = \{(a,b)$: for some $l \in \{0,1\}$, $(a',b') \in R(N_l)$, $M' \models a' \leqq a \leqq b \leqq b'\}$.*

(2) *For every candidate $\bar{M}$ there is $\bar{M}'$, $\bar{M} \leqq^{w} \bar{M}'$ such that*

(i) *there are atoms $c_0, \ldots, c_{k-1}$ of M'_0 ($k = k(\bar{M}')$) such that $a_l < c_l$ ($l < k$) and $M'_l = \langle M'_0, a'_0, \ldots, a'_{l-1} \rangle$,*

(ii) $k(\bar{M}) = k(\bar{M}')$, $M_k = M'_k$.

(3) *If $M \in K$, $c < b \in M$, M is in K, then there are a, N, c', b', such that $R(M,N,a), N \models$ "$c < c' < b' < b \wedge R(c',b')$".*

(4) *If $M \in K$, there are a, N such that $R(M,N,a), N \models P(a)$.*

PROOF. (1) Easy

(2) Just like 2.4.

(3) N, as a B.A., is freely generated by M, c', b', except the relations $c \leqq c' \leqq b' \leqq b$.

(4) N, as a B.A., is freely generated by N, a (remember $M \models \neg R(0,1)$).

CLAIM 3.5. *Let $\bar{M}$ ($k = k(\bar{M})$) be a candidate, $c_0, \ldots, c_{k-1}$ atoms of M_0, $a_l < c_l$, $M_l = \langle M_0, a_0, \ldots, a_{l-1} \rangle$. Suppose $\bar{M}^m$ ($m < n$) are copies of $\bar{M}$ over M_0, and $M^m_k \subseteq N$ and*

(i) $P(N) = \bigcup_{m<n} P(M^m_k)$,

(ii) $R(N) = \{(a',b')$: *for some* m, $(a,b) \in R(M^m_k)$, $N \models a \leqq a' \leqq b' \leqq b\}$,

(iii) *for any* $l < k$, $m < n$, $a^m_l \notin \langle c_l, a^0_l, \ldots, a^{m-1}_l \rangle$,

(iv) *for any* $l < k$, $m(1), m(2) < n$, $a_l^{m(1)} \cap a_l^{m(2)} \neq 0$,

$$(c_l - a_l^{m(1)}) \cap (c_l - d_l^{m(2)}) \neq 0.$$

Then *condition* (*) *of* 1.9 *holds.*

PROOF. Left to the reader.[†]

CLAIM 3.6. *Let B be a finite Boolean algebra and $J \subseteq B$ is such that*:

(1) $\bigcap_{a \in J} a \neq 0$, $\bigcup_{a \in J} a \neq 1$,

(2) $J = \{a_m : m < n\}$ *is such that* $a_l \notin \langle a_0, \dots, a_{l-1} \rangle$.

Let M^* be as in 1.7 (*for our K, R*).

Then for any uncountable $I \subseteq B$, there is an embedding of B into M^* (*as a Boolean algebra*) *such that J is mapped into I.*

PROOF. By 3.5.

PROOF OF THEOREM 3.3. Immediate by 1.7, 1.9 and the previous claims.

§4. K-Inevitable models

DEFINITION 4.1. For a class K of models and cardinal λ, M is called K-λ-*inevitable* if M can be embedded in every $N \in K$, $\|N\| \geqq \lambda$. For $\lambda = 0$ we omit it.

REMARK. Note that M may be not in K.

CLAIM 4.1. *If M is K-λ-inevitable, $K' \subseteq K$, $\lambda' \geqq \lambda$, $M' \subseteq M$* then *M' is K'-λ'-inevitable.*

CLAIM 4.2. *Suppose K is very nice. Then*:

(1) *K^H has, up to isomorphism, a unique countable model $M_{\aleph_0}$ which is not finitely generated,*

(2) *$M_{\aleph_0}$ is homogeneous and universal* (*for K^1*),

(3) *$M_{\aleph_0}$ is K^H-$\aleph_0$-inevitable.*

DEFINITION 4.3. For K nice and R K-nice, let

$$K^H_{\aleph_1}[R] = \{M \in K^H_{\aleph_1} : \text{for every } M_0 \in K,\ N_0 \in K \text{ and } a \text{ such that } R(M_0, N_0, a)$$
$$\text{and } N,\ M_0 \subseteq N \subseteq M,\ \|N\| \leqq \aleph_0$$
$$\textit{there are } N', N''\ N \subseteq N' \subseteq N'' \subseteq M,$$

[†] Condition (iv) of 3.5 is needed to show "$R(x,x) \rightarrow \neg P(x)$" holds in N.

$\|N''\| \leqq \aleph_0$, and an embedding f of N_0 into N'' over M_0 such that $R(N', N'', f(a))$

CLAIM 4.4. *Suppose K is very nice and R K-nice. Let*

$$R' = \{(M, N, a): \textit{for some } N',\ R(M, N', a),\ M \subseteq N' \subseteq N, \textit{ and } M, N, N' \textit{ are finitely generated}\}.$$

Then R' is K-very nice and $K^H_{\aleph_1}[R] = K^H_{\aleph_1}[R']$.

PROOF. Easy.

THEOREM 4.5. *Assume $\diamondsuit(\aleph_1)$.*

If K is very nice and R K-nice then the following are equivalent:

(a) *There is an uncountable $K^H_{\aleph_1}[R]$-inevitable model.*

(b) *There is a candidate $\bar{M}$ and $b \in \mathrm{dom}(\bar{M}) - M_0$, $k = k(\bar{M})$ such that for every n:*

$(*)_n$ *Suppose N_i, N'_i $(i \leqq nk)$, f_m $(m \leqq n)$ are such that: $N_i \subseteq N'_i \subseteq N_{i+1}$, f_m maps M_l into N_{mk+l}, $f_m \restriction M_0 =$ the identity, and $R(N'_{mk+l}, N_{mk+l+1}, f_m(a_l))$. Then the isomorphism type of $(\langle f_0(b), \ldots, f_n(b)\rangle_{N_{nk}}, f_0(b), \ldots, f_n(b))$ is determined by $\bar{M}, b$ and n and $f_m(b) \notin N_0$.*

PROOF. By Claim 4.3 we can assume R is K-very nice, and in $(*)$ of 4.5 let $N_i = N'_i$.

(b) $\Rightarrow$ (a)

Let M^* be in $K^H_{\aleph_1}[R]$.

Let $\bar{M}$ be as in (b), and w.l.o.g. $M_0 \subseteq M^*$. We define by induction on $\alpha < \omega_1$, models $N'_\alpha \subseteq M^*$, $N'_{\alpha,l}$ $(l < k)$ and embeddings f_α of $\mathrm{dom}(\bar{M})$ into M^* such that:

(i) $\|N_\alpha\| \leqq \aleph_0$; $\alpha < \beta \Rightarrow M_0 \subseteq N_\alpha \subseteq N_\beta$ and for δ limit $N_\delta = \bigcup_{\alpha<\delta} N_\alpha$,

(ii) $N_\alpha = N_{\alpha,0} \subseteq N'_{\alpha,0} \subseteq N_{\alpha,1} \subseteq N'_{\alpha,1} \subseteq \cdots \subseteq N_{\alpha,k} = N'_{\alpha,k} \subseteq N_{\alpha+1}$,

(iii) f_α embed M_l into $N_{\alpha,l}$ and $R(N'_{\alpha,l}, N_{\alpha,l+1}, f_\alpha(a_l))$.

This is done by induction on α. If we have defined for every $\beta < \alpha$, it is easy to define N_α (M_0 for $\alpha = 0$, $N_{\alpha-1,k}$ for α successor $\bigcup_{\beta<\alpha} N_\beta$ for α limit). Now we define $f_\alpha \restriction M_l$, $N_{\alpha,l}$, $N'_{\alpha,l}$ by induction on l. Remember $f_\alpha \restriction M_0$ is the identity. $N_{\alpha,0} = N_\alpha$. If $f_\alpha \restriction M_\alpha$, $N_{\alpha,l}$ are defined as $M^* \in K^H_{\aleph_1}[R]$ there are $N'_{\alpha,l}$, $N_{\alpha,l+1}$, $f_\alpha \restriction M_{l+1}$ as required (in (ii), (iii)) (see Definition 4.3).

For $l = k$, $N'_{\alpha,k} = N_{\alpha,k}$.

Let $b_\alpha = f_\alpha(b)$. Now by (b), for every $\alpha(0) < \cdots < \alpha(n) < \omega_1$, the isomorphism type of $(\langle b_{\alpha(0)}, \ldots, b_{\alpha(n)}\rangle_{M^*}, b_{\alpha(0)}, \ldots, b_{\alpha(n)})$ depends on $\bar{M}$, b and n only. So the submodel of M^* generated by $\{b_\alpha : \alpha < \omega_1\}$ is $K^H_{\aleph_1}[R] - \aleph_1$-inevitable.

REMARK. Note $\{b_\alpha : \alpha < \omega_1\}$ is an indiscernible sequence, i.e., for any $\alpha(0) < \cdots < \alpha(n) < \omega_1$, $\beta(0) < \cdots < \beta(n) < \omega_1$ there is an isomorphism f from $\langle b_{\alpha(0)}, \ldots, b_{\alpha(n)} \rangle_{M^*}$ onto $\langle b_{\beta(0)}, \ldots, b_{\beta(n)} \rangle_{M^*}$, $f(b_{\alpha(l)}) = b_{\beta(l)}$.

(a) $\Rightarrow$ (b)

We use 1.7 to build a pair of models in $K^H_{\aleph_1}[R]$, with a f.g. approximation being a pair of f.g. approximations. Formally let the language of K, L be $\{P_i, F_j : i < i_0, j < j_0\}$. Let $L^l = \{P^l_i, F^l_j : i < i_0, j < j_0\}$ (for $l = 1,2$) be disjoint copies of L. For $M \in K$, $l \in \{1,2\}$ let $M^{[l]}$ be an L^l-copy of M, i.e. $|M^{[l]}| = |M|$, $(P^{[l]}_i)^{M^{[l]}} = P^M_i$, $(F^{[l]}_i)^{M^{[l]}} = F^M_i$. Let $K^l = \{M^{[l]} : M \in K\}$.

Now we define L^p, K^p:

$$L^p = L^1 \cup L^2 \cup \{P_1, P_2, F_1, F_2\},$$

P_1, P_2 monadic predicates, F_1, F_2 monadic function symbols.[†]

$$K^p = \{M : M \text{ an } L^p\text{-model}, P^M_1, P^M_2, \text{are disjoint; for } l = 1,2:$$

$$(M \restriction P^M_l) \restriction L^l \in K^l \qquad F^{[l]}_j(x_1, \ldots, x_n) = x_1 \text{ if } \{x_1, \ldots, x_n\} \not\subseteq P^M_l,$$

$$F_l \restriction (P^M_1 \cup P^M_2) \text{ is the identity and for } x \in |M| - P^M_1 \cup P^M_2,\ F_l(x) \in P^M_l\},^{\dagger\dagger}$$

and we define R^p, $(M, N, a) \in R^p$ *iff* $M \in K^p$, $N \in K^p$, $a \in |N| - P^M_1 \cup P^M_2$ and for $l = 1,2$, $R((M \restriction P^M_l) \restriction L^l, (N \restriction P^N_l) \restriction L^l, F_l(a))$. Easily K^p is very nice and R^p is nice. Let M^* be as in 1.7 for K^p, R^p. Let $M^*_l = (M^* \restriction P^{M^*}_l) \restriction L^l$; we can prove $M^*_l \in K^H_R[\aleph_1]$. We shall prove that no uncountable member of K is embeddable to both M^*_1, M^*_2. If g_l embed N into M^*_l, $\|N\| = \aleph_1$, $N \in K^1$, let $a_i \in N$ $(i < \omega_1)$ be distinct and let $I = \{(g_1(a_i), g_2(a_i)) : i < \omega_1\}$. So by 1.7(b) there is a witness for (M^*, I), say $(\bar{M}, \bar{b})$ $\bar{b} = \langle b^0, b^1 \rangle$ (so $b^l \in P_l$). We can naturally define $\bar{M}^l, b^l$ for $l = 1,2$. If $(\bar{M}^1, \bar{b}^1)$ does not satisfy $(*)$ of (b), and we have distinct ways to amalgamate, then we shall get a contradiction. As we have said in the beginning of the proof, R is very nice so $(*)_n$ is simplified (i.e. $N^1_l = N_l$). Let $\bar{M} = \langle M_k, \ldots, M_1, M_0, c_{k-1}, \ldots, c_0 \rangle$.

So suppose that for $\alpha = 0,1$, we have n, and $N_{i,\alpha}$ for $i \leqq nk$ (where $k = k(\bar{M})$) such that:

(a) $N_{i,\alpha} \subseteq N_{i+1,\alpha}$, $f_{m,\alpha}$ an embedding of M^1_l into $N_{mk+l,\alpha}$, $f_{m,\alpha} \restriction M^1_0 =$ the identity and $R(N_{mk+l,\alpha}, N_{mk+l+1,\alpha}, f_m(c_l))$ holds.

(b) There is no isomorphism from $\langle f_{0,0}(b_1), \ldots, f_{n-1,0}(b_1) \rangle_{N_{nk,0}}$ (as submodel $N_{n-k,0}$) onto $\langle f_{0,1}(b_1), \ldots, f_{n-1,1}(b_1) \rangle_{N_{nk,1}}$ which maps $f_{l,0}(b^1)$ to $f_{l,1}(b^1)$ for $l < n$.

[†] If we allow in §1 $R(M, N, \bar{a})$ for $\bar{a}$ a finite sequence, K^P will be simplified.

[††] Of course, every $M \in K^p$ is required to be finitely generated so $|M| - P^M_1 \cup P^M_2$ is a finite set.

We can also find $N_{i,2} \in K (i \leqq nk)$ and $f_{m,2}$ (for $m \leqq mk$) such that $f_{m,2}$ embedded M_l^2 into $N_{mk+l,2}$ and $R(N_{mk+l,2}, N_{mk+l+1,2}, f_m(c_l))$ holds. W.l.o.g. there is no isomorphism from $\langle f_{0,1}(b^1), \ldots, f_{n-1,1}(b^1)\rangle_{N_{nk,1}}$ onto $\langle f_{0,2}(b^2), \ldots, f_{n-1,2}(b^2)\rangle_{N_{nk,2}}$ which maps $f_{l,1}(b^1)$ to $f_{l,2}(b^2)$ [otherwise interchange $N_{i,1}$, $f_{m,1}$ with $N_{i,0}$, $f_{m,0}$].

By Definition 1.4 w.l.o.g. $N_{i,1} \in K$ (in fact we could do this in $(*)_n$). W.l.o.g. $N_{i,1}$, $N_{i,2}$ are disjoint.

We can now define M_i^*, $(M_i \restriction P_l^{M_l}) \restriction L_l = N_{i,l}$ for $l = 1,2$ and $a_i^* \in M_{i+1}^*$ is such that $F_l(a_i^*) = f(a_i)$ for $l = 1,2$.

So we can find $i_0 < i_1 < \cdots < i_{n-1}$ such that for $l = 1,2$ there is an embedding g_l of $\langle f_{0,l}(b^l), \ldots, f_{n-1,l}(b^l)\rangle_{N_{nk,l}}$ into $(M^* \restriction P_1) \restriction L_1$, $g_l(f_{m,l}(b^l)) = g_l(a_{i_m})$. We get now a contradiction easily.

Claim 4.6. *For any nice K, there is a maximal K-nice, R, i.e. for every other K-nice R', $R' \subseteq R$. In fact $R(M,N,a)$, $M' \subseteq M \subseteq N \subseteq N' \in K$ implies $R(M',N',a)$ so R is K-very nice.*

Proof. Define inductively R_α:

$R_0 = \{(M,N,a) : M \subseteq N,\ a \in N - M,\ M \in K,\ N \in K\}$,

$R_{\alpha+1} = \{(M,N,a) : R_\alpha(M,N,a)$ and for every $M' \in K$, $M \subseteq M'$, there are N', f such that $M' \subseteq N'$, f is an embedding of N into N' over M, $f(a) \notin M'$ and $R_\alpha(M',N,f(a))$ holds$\}$,

$$R_\delta = \bigcap_{\alpha<\delta} R_\alpha \text{ for } \delta \text{ limit.}$$

Clearly R_α is decreasing, so as K has, up to isomorphism, only $\aleph_0$-members, each countable and finitely generated, for some $\alpha < \omega_1$, $R_\alpha = R_{\alpha+1}$ hence $\beta < \alpha \Rightarrow R_\alpha = R_\beta$. Now R_α is as required.

Now we turn to a specific application.

Claim 4.7.[†] *Let G be a countable existentially closed group*

$$K(G) = \{M : M \text{ a finitely generated subgroup of } G\},$$

$$R(G) = R_T(K).$$

Then $K(G)$ is very nice, $R(G)$ K-very nice, and $K(G)^H_{\aleph_1} = K(G)^H_{\aleph_1}[R(G)]$.

[†]This is known; the way to prove it is by Zeigler's theorem: if G is an existentially closed group, P a recursive set of equations and inequation with finitely many parameters which is realized in some group extending G, then it is realized in G.

THEOREM 4.8. *Assume* $\diamondsuit(\aleph_1)$.

For G countable existentially closed, there is no uncountable $K(G)^H$-$\aleph_1$-*inevitable model.*

PROOF. We shall use Theorem 4.5, of course. So let a candidate $\bar{M}$ and $b \in M_k - M_0$ ($k = k(\bar{M})$) be given. We shall contradict $(*)_3$ from 4.5(b). This we do in two steps. So let $(\bar{M}^l, b^l)$ ($l = 0,1,2$) be three copies of $(\bar{M}, b)$ over M_0. $g_l : M \to \bar{M}_l$ is isomorphic. Now first we show there are two possibilities to amalgamate the M_k^l ($l = 0,1,2$) over M_0 as groups, so that $\langle b^0, b^1, b^2 \rangle$ are different. The second step is to show we can have such amalgamation in $K(G)$.

The second step is as in the proof of 4.7.

So we concentrate on the first step.

Take the free product of M_k^0, M_k^1, M_k^2 over M_0 and call it N^*. For $\{l, l(1), l(2)\} = \{0,1,2\}$ let N_l be the normal subgroup of N^* generated by $\{g_{l(1)}(c)^{-1} g_{l(2)}(c) : c \in M_k\}$.

Notice N^*/N_0 is not good as an amalgamation because $g_2(a_0) \in \langle g_0(M_k) \cup g_1(M_k) \rangle$. Similarly N^*/N_1 is not. But $N^*/N_0 \cap N_1$ is good as an amalgamation.

Now in N^*, $[b_0 b_2^{-1}, b_1 b_2^{-1}]$ is not the unit (being free amalgamation) ($[x,y]$ is $xyx^{-1}y^{-1}$, the commutator) but $b_0 b_2^{-1} \in N_1$, $b_1 b_2^{-1} \in N_0$ so $[b_0 b_2^{-1}, b_1 b_2^{-1}] \in N_0 \cap N_1$, hence in $N^*/N_0 \cap N_1$ it is the unit.

REMARK. This kind of proof of non-uniqueness can be carried out for many classes (we then have to replace "the normal subgroup generated by..." by "the congruence relation generated by $\{g_{l(1)}(c) \equiv g_{l(2)}(c) : c \in M_k\}$".

§5. Banach spaces

Our result in this section is the following theorem. Note that 5.1(A) solved a problem of Davis and Johnson [2] and was announced in Abstracts of the Am. Math. Soc. 5.1(C) gave another proof of a result of [8] (but the example there has some additional properties), and 5.1(B) solves a problem.

I would like to thank Johnson for explaining to me the problem from [2] and how to clean up the proof from computations.

THEOREM 5.1. *Assume* $\diamondsuit(\aleph_1)$.

There is a non-separable Banach B *such that*:

(A) *Among any* $\aleph_1$ *elements, one belongs to the closure of the convex hull of the others.*

(B) *For every non-separable closed subspace* B_1 *of* B, B/B_1 *is separable.*

(C) *For any* (*linear bounded*) *operator* T *from* B *to* B, *there are a real number* c

and a (linear bounded) operator T_1 with separable range, such that $T = cI + T_1$ (I is the identity operator).

REMARK. Note that $B = \bigcup_{i<\omega_1} B_i$, B_i increasing, continuous and separable, and each B_i is a Gurari space. Also B is l_1-predual.

The theorem is proved by the following lemma, which contains more numerical information. We can easily give more similar conclusions to 5.2, and even phrase the general case (just look at the proof of 5.2(4)).

LEMMA 5.2. *There is a non-separable Banach space B such that:*

(1) *For any $y_i \in B$ $(i < \omega_1)$ and $\varepsilon > 0$ there are $i < j$ such that*

$$\| y_i - y_j/2 \| \leqq \| y_i \|/2 + \varepsilon.$$

(2) *For any $y_i \in B$ $(i < \omega_1)$ and $\varepsilon > 0$ and $n > 0$ there are $i(0) < \cdots < i(n) < \omega_1$ such that*

$$\| y_{i(0)} - (y_{i(1)} + y_{i(2)} + \cdots + y_{i(n)})/n \| \leqq \| y_{i(0)} \|/n + \varepsilon.$$

(3) *If y_i, z_i $(i < \omega_1)$ are in B, $\| y_i \| = 1 = \| z_i \|$ $\| z_i - z_j \| \geqq 1 - 1/n$ (for $i \neq j < \omega_1$), $\varepsilon > 0$,* then *there are $i(1) < \cdots < i(n) < i(n+1) < \cdots < i(3n+1)$ such that*

$$\left\| y_{i(n+1)} - \left(\sum_{l=1}^{n} y_{i(l)} \right) \Big/ n + \left(\sum_{l=1}^{2n} (-1)^l z_{i(n+1+l)} \right) \Big/ n \right\| \leqq 1/n + \varepsilon.$$

(4) *If y_i, z_i $(i < \omega_1)$ are in B, $\| y_i \| = 1$, c a rational number and for $i < j$, $d(z_j, \langle y_i, z_j, y_j \rangle) \geqq c$ ($\langle y_i, z_j, y_j \rangle$ is the subspace $\{y_i, z_j, y_j\}$ generate),* then *for any even n and $\varepsilon > 0$ there are $i(0) < \cdots < i(n+1)$, such that*

$$\left\| \sum_{l=0}^{n-1} (-1)^l y_{i(l)} \right\| \leqq 1 + \varepsilon, \qquad \left\| \sum_{l=0}^{n-1} (-1)^l z_{i(l)} \right\| \geqq cn/2 - \varepsilon.$$

PROOF OF THEOREM 5.1 FROM LEMMA 5.2.

5.1(A). Let $y_i \in B$ $(i < \omega_1)$; if no y_i belong to the closure of the convex hull of $\{y_j : j \neq i\}$, then for every i for some $n(i) > 0$, the distance from y_i to the closure of the convex hull of $\{y_j : j < i\}$ is $> 1/n(i)$. As there are only countably many $n(i)$'s, for some n, $A = \{i < \omega_1 : n(i) = n\}$ is uncountable. Now $\{y_i : i \in A\}$ does not satisfy 5.2(2) for $n+1$, contradiction. (Note that we prove that some i belongs to the closure of the convex hull of $\{y_j : i < j < \omega_1\}$.)

5.1(B). Let $B_1 \subseteq B$ be a closed non-separable subspace, with B/B_1 non-separable. Choose $n > 2$, and then choose by induction on $i < \omega_1$, $z_i \in B_1$, so that $\| z_i \| = 1$,

$$d(z_i, \langle z_j : j < i \rangle) \geqq 1 - 1/n$$

(where $\langle A \rangle$ is the linear span of A). Next choose by induction on $i < \omega_1$, $y_i \in B$, such that $\| y_i \| = 1$,

$$d(y_i, \langle B \cup \{y_j : j < i\}\rangle) \geqq 1 - 1/n$$

(possible by the non-separability of $B_1, B/B_1$ resp.). Now use 5.2(3), and get a contradiction to the choice of $y_{i(n+1)}$.

5.1(C). For any $B_1 \subseteq B$ let $c(T, B_1) = \sup\{d(Tx, \langle B_1, x\rangle) : x \in B, \| x \| = 1\}$. If for every separable B_1, $c(T, B_1) > 0$ we get contradiction by 5.2(4), and if for some separable B, $c(T, B_1) = 0$, we can prove 5.1(C).

PROOF OF LEMMA 5.2. We shall prove that there is a vector space M^* over the rationals, with a norm, which satisfies the conclusions of 5.2, moreover without the ε. It is easy to check that the completion of M^* is as required.

Now we define K as the family of finite-dimensional vector spaces M over the rationals, which are also norm spaces, where the norm has the form

$$\| x \| = \text{Max}\{|f(x)| : f \in F\}$$

where $F = F_M$ is a finite set of (linear) functionals, from M to the rationals, and w.l.o.g. $\| f \| = 1$ (by the definition $\| f \| \leqq 1$, and if $\| f \| < 1$ we can omit it without changing the norm). In the proof a functional from M means a rational one.

FACT a. K is a very nice family.

Being a nice family is trivial. For the "very" we first have to find a joint embedding to $M_0, M_1 \in K$. Let $N = M_0 \times M_1$ (direct product as vector spaces), for any functional f_0, f_1 of M_0, M_1 resp. $\langle f_0, f_1 \rangle$, defined by $\langle f_0, f_1\rangle(x, y) = f_0(x) + f_1(y)$, be a functional of $M_0 \times M_1$ with rational values. We let

$$F_N = \{\langle f_0, f_1 \rangle : f_0 \in F_{M_0}, f_1 \in F_{M_1}\}.$$

So by F_N, N is made into a norm space, and the obvious embedding completes the construction.

We can do a similar construction for amalgamation M_0, M_1 over M. Let M^* be the subvector-space $\{(x, -x) : x \in M\}$ of $M_0 \times M_1$; let $N = M_0 \times M_1 / M^*$. By the Hahn–Banach theorem, for every functional g of M, $\| g \| \leqq 1$ and $l = \{1,2\}$, g has an extension $g[M_l]$ to a functional of M_l of norm $\leqq 1$; we can assume that $g = 0$ implies $g[M_l]$ is 0. Note also that any functional f of $M_0 \times M_1$ which is zero on M^*, defines naturally a functional f/M^* of N with the same norm.

We let

$$F_N = \{\langle f, (f \restriction M)[M_1]\rangle / M^* : f \in F_{M_0}\} \cup \{\langle (f \restriction M)[M_0], f\rangle / M^* : f \in F_{M_1}\}.$$

We leave the details to the reader.

We shall use the notation here later.

Now let

$$R = \{(M, N, a) : M \subseteq N \text{ both in } K,\ a \in N \text{ and for some functional } f \text{ of } N, \|f\| = 1 = \|a\|, f \restriction M = O_M \text{ and } \|a\| = 1\}.$$

FACT b. R is K-very nice.

The only part of Definition 1.2 which is not totally trivial is 1(c), so let $M \subseteq M' \in K$, $(M, N, a) \in R$. We let $M^* = \{(x, x) : x \in M\}$, $N = M' \times N / M^*$, and as w.l.o.g. the functional f such that $f(a) = 1 = \|a\|$, $f \restriction M = O_M$ is in F_N, and as $(f \restriction M)[N] = 0$ (because $f \restriction M = 0$), we can act as above.

So let M^* be the model for which Theorem 1.7 asserts its existence. We shall prove that M^* is as required in Lemma 5.2. In each case we have an uncountable set I of elements or pairs, and by 1.7 it has a witness. Using it we prove the requirements of 5.2.

In all cases we are given a witness $(\bar{M}, \bar{b})$ for I. We let $k = k(\bar{M})$,

$$N^0_m = M^m_k = M_k \times M_k \times \cdots \times M_k \quad (m \text{ times}),$$

$$M^+_m = \{(q_1 x, \ldots, q_m x) : x \in M_0, q_1, \ldots \text{ rationals}, \Sigma^m_{i=1} q_i = 0\},$$

$$N_m = N^0_m / M^+_m.$$

Let $H_i : M_k \to M^m_k$, $H_i(x) = (0, \ldots, 0, x, 0, \ldots, 0)$ ($i - 1$ zeros, x, $m - i$ zeros) are the natural embeddings we shall use.

We can assume F_{M_k} is such that $O_{M_k} \in F_{M_k}$, and for any $i < k$, there is $f_i \in M_k$, $f_i(a_i) = 1 = \|a_i\|$, $f_i \restriction M_i = O_{M_i}$.

Now the crux of the matter will be to define a finite set F of functionals from M^m_k to Q, which are zero on M^+_m, and define for $x \in N_m$

$$\|x\| = \operatorname{Max}\{(f \restriction M^+_m)(x) : f \in F\}.$$

What are the requirements on F?

Requirement 1: For every $f \in F$ and $i \in [1, m]$ for some functional f' of M_k of norm $\leqq 1$, $H_i f' = f \restriction \operatorname{Range}(H_i)$.

This guarantees that $\|H_i(x)\| \leqq \|x\|$.

Requirement 2: For every $f' \in F_{M_k}$ and $i \in [1, m]$ for some $f \in F$, $H_i f' = f \restriction \operatorname{Range}(H_i)$.

This guarantees $\|H_i(x)\| \geqq \|x\|$, so, together with Requirement 1, it is shown that the H_i's are embeddings.

Requirement 3: For every $i \in [2, m]$, $l \in [0, k-1]$ there is $f_{i,l} \in F$, $f_{i,l}(H_i(a_i)) = 1$, $f_{i,l}(H_i(x)) = 0$ for $x \in M_l$ and $f_{i,l}(H_j(x)) = 0$ for $x \in M_k$, $j < i$.

This guarantees that we get a candidate.

Why do we ignore $i = 1$? Because Requirement 2 takes care of it anyhow.

Note that Requirements 2 and 3 force us to put some functionals into F, whereas Requirement 1 restricts what functionals we may put into F.

For functionals f_i of M_k of norm $\leqq 1$, $f_i \restriction M_0$ fixed, let $\bar{f} = \langle f_1, \dots, f_m \rangle$ be the following function on M_k^m:

$$\bar{f}(x_1, \dots, x_m) = \sum_{i=1}^{m} f_i(x_i)$$

(we do not distinguish strictly between $\bar{f}$ and $\bar{f}/M_m^+$). Clearly $\bar{f}$ is zero on M_m^+.

Assumption 4: We shall use only functionals of this form, and this guarantees Requirement 1.

A particular case is $\bar{f} = \langle f, \dots, f \rangle$, $f \in F_{M_k}$. We always put all those functionals into F, thus guaranteeing Requirement 2. The set of all such F will be denoted by F_0, and $F_1 = \{f_{i,l} : i \in [2, m], l \in [0, k-1]\}$ and $F_2 = F - F_0 \cup F_1$. So F_1 guarantees Requirement 3.

Now we get to the specific cases, corresponding to the parts of 5.2.

PROOF OF 5.2(1) FOR M^*. So $I \subseteq M^*$ is uncountable, $(\bar{M}, b)$ a witness for it. We let $m = 2$, $F_3 = \varnothing$

$$\bar{f}_{2,l} = \langle 0, f_l \rangle.$$

We want to prove $\|H_1(b) - H_2(b)/2\| \leqq \|H_1(b)\|/2$. It is enough to check that for every $\bar{f} \in F$, $|\bar{f}(H_1(b) - H_2(b)/2)| \leqq \|b\|/2$.

Let $\bar{f} \in F_1$. Clearly $\bar{f} = \langle g, g \rangle$,

$$\bar{f}(H_1(b) - H_2(b)/2) = \bar{f}(b, -b/2) = g(b) - g(b/2) = g(b/2),$$

so its absolute value is $\leqq \|g\| \|b/2\| = \|b\|/2$, as required.

If $\bar{f} \in F$, $\bar{f} \notin F_1$ then $\bar{f} = \bar{f}_{2,l}$ for some l. Hence let $\bar{f}_{2,l} = \langle 0, f_l \rangle$. Therefore

$$\bar{f}(H_1(b) - H_2(b)/2) = \bar{f}(b, b/2) = 0 + f_l(-b/2) = -f_l(b)/2,$$

so its absolute value is $\leqq |f_l(b)|/2 = \|f_l\| \|b\|/2 \leqq \|b\|/2$.

PROOF OF 5.2(2) FOR M^*. We let $m = n+1$, $F_3 = \varnothing$, and let

$$\bar{f}_{i,l} = \langle g^1_{i,l}, \dots, g^m_{i,l} \rangle,$$

$$g^j_{i,l} = \begin{cases} O_{M_k}, & i \neq j, \\ f, & i = j. \end{cases}$$

Let $\tau = H(b_1) - (H(b_2) + \cdots + H(b_m))/n$. We want $\|\tau\| \leqq \|b_1\|/n$, i.e., for every $\bar{f} \in F$, $|\bar{f}(\tau)| \leqq \|b_1\|/n$.

For $\bar{f} \in F_0$, clearly $\bar{f}(\tau) = 0$, and for $\bar{f} \in F_2$, $\bar{f} = \bar{f}_{i,l}$ and $i > 1$, so

$$\bar{f}(\tau) = g^1_{i,l}(b) - \left(\sum_{j=2}^{m} g^j_{i,l}(b) \right) \Big/ n = 0 - g_l(b)/n.$$

So $|\bar{f}(\tau)| \leqq |g_l(b)|/n \leqq \|g_l\| \|b\|/n \leqq \|b\|/n$, as required.

PROOF OF 5.2(3) FOR M^*. Let $(\bar{M}, \langle y, z \rangle)$ be the witness.

First we show that $d(z, M_0)$ (the distance from z to M_0) is $\geqq 1 - 1/n$. Otherwise let $m = 2$, and define F as in the proof of 5.2(1) for M^*, so there necessarily $\|H_1(z) - H_2(z)\|$ is $|\bar{f}_{2,l}(H_1(z) - H_2(z))| = |f_l(z)|$ for some l, but as $f_l \restriction M_0 = O_{M_0}$ clearly

$$|f_l(z)| \leqq \|f_l\| d(z, M_0) = d(z, M_0).$$

But by a hypothesis $\|z_i - z_j\| \geqq 1 - 1/n$ for $i \neq j$, so as $(\bar{M}, \langle y, z \rangle)$ is a witness for $\{\langle y_i, z_i \rangle : i < \omega_1\}$

$$\|H_1(z) - H_2(z)\| \geqq 1 - 1/n.$$

So we have really shown $d(z, M_0) \geqq 1 - 1/n$.

Obviously $\|y\| = \|z\| = 1$ for even more trivial reasons.

Now let $m = 3n+1$, and define F so that $\tau = \tau_1 - \tau_2 + \tau_3$ has norm $\leqq 1/n$ where

$$\tau_1 = H_{n+1}(y), \quad \tau_2 = \sum_{l=1}^{n} H_l(y) \Big/ n, \quad \tau_3 = \left(\sum_{l=1}^{2n} (-1)^l H_{n+1+l}(z) \right) \Big/ n.$$

We let $F_2 = \varnothing$ and $f_{i,l} = \langle g^1_{i,l}, g^2_{i,l}, \dots, g^m_{i,l} \rangle$ be defined as:

Case α: $i \neq n+1$

$g^j_{i,l}$ is f_l if $i = j$, and O_{M_k} otherwise.

Case β: $i = n+1$

As $d(z, M_0) \geqq 1 - 1/n$, there is a rational functional g of M_k, $g \restriction M_0 = O_{M_0}$, $g(z) = 1 - 1/n$, $\|g\| \leqq 1$. Let

$$g^j_{i,l} = \begin{cases} 0, & 1 \leqq j \leqq n, \\ f_l, & j = n+1, \\ 0, & j > n+1, \quad j-(n+1) \text{ even}, \\ f_l(y)\cdot g, & j > n+1, \quad j-(n+1) \text{ odd}. \end{cases}$$

Is $f_{i,l}$ as required? (See Requirement 3.) Clearly it is zero in the places required, but is $g^j_{i,l}$ as required in Assumption 4? The only doubtful case is $f_l(z)g$, but g is rational of norm $\leqq 1$, and $f_l(y)$ is a rational number, and as $\| y \| = 1$ (see above) and f_l has norm $\leqq 1$ clearly $|f_l(y)| \leqq 1$, so $f_l(y)$ is as required.

So let us check that $\|\tau\| \leqq 1/n$, i.e., for every $\bar{f} \in F$, $|\bar{f}(\tau)| \leqq 1/n$. For $\bar{f} \in F_0$, $\bar{f}(\tau) = 0$ so we have no problem. So we have to check $f_{i,l}$ only.

If $i \neq n+1$, case α holds, so

$$f_{i,l}(\tau_1 - \tau_2 + \tau_3) = f_{i,l}(\tau_1) - f_{i,l}(\tau_2) + f_{i,l}(\tau_3).$$

Now $f_{i,l}(\tau_1) = 0$ as $i \neq n+1$, and from the others only one term is non-zero, $g^j_{i,l}(y)/n$ or $\pm g^j_{i,l}(z)/n$, so as $\|g^j_{i,l}\| \leqq 1$, $\|y\| = \|z\| = 1$ we finish this case.

We are left with the case $i = n+1$. So

$$\begin{aligned} f_{i,l}(\tau) &= f_{i,l}(\tau_1) - f_{i,l}(\tau_2) + f_{i,l}(\tau_3) \\ &= f_l(y) - 0 + \sum_{\substack{l=1 \\ l \text{ even}}}^{2n} 0 + \sum_{\substack{l=1 \\ l \text{ odd}}}^{2n} (-1)^l f_l(y) g(z)/n \\ &= f_l(y)(1 - g(z)) = f_l(y)(1-(1-1/n)) = f_l(y)/n. \end{aligned}$$

Hence

$$|f_{i,l}(\tau)| \leqq |f_l(y)|/n = \|f_l\| \, \|y\|/n = 1/n$$

as required.

Proof of 5.2(4) for M^*. So let $(\bar{M}, \langle y, z\rangle)$ be a witness for I. Just as in the beginning of the proof of 5.2(3), $d(z, \langle M_0, y\rangle) \geqq c$.

We let $m = n$, $f_{i,l} = \langle g^1_{i,l}, \ldots, g^m_{i,l}\rangle$, $g^j_{i,l}$ is f_l if $i = j$ and zero otherwise. In this case F_3 is non-empty; it is $\{\bar{g}\}$, $\bar{g} = \langle g_1, \ldots, g_n\rangle$, where g_l is zero for l odd, and is g for l even, where g is a rational functional of M_k of norm $\leqq 1$ which is zero on $\langle M_0, y\rangle$, and $g(z) = c$ (exists as $d(z, \langle M_0, y\rangle) \geqq c$).

Clearly $F = F_0 \cup F_1 \cup F_2$, and we only have to check that

$$\tau_1 = \left(\sum_{l=1}^{m} (-1)^{l+1} H_l(y) \right) \quad \text{and} \quad \tau_2 = \left(\sum_{l=1}^{m} (-1)^{l+1} H_l(z) \right)$$

gets the right norm.

For $\bar{f} \in F_0$, $\bar{f}(\tau_1) = \bar{f}(\tau_2) = 0$ and for $\bar{f} \in F_1$, $\bar{f}(\tau_1) = (-1)^{l+1} f_l(y)$, $\bar{f}(\tau_2) = (-1)^{l+1} f_l(z)$ for some l, so clearly for $\bar{f} \in F_0 \cup F_1$

$$|\bar{f}(\tau_1)| \leqq \| y \| = 1, \qquad |\bar{f}(\tau_2)| \leqq \| z \| / n.$$

Now for $\bar{f} \in F_3$, clearly $\bar{f}(\tau_1) = 0$, so we can conclude that $\|\tau_1\| \leqq 1$. On the other hand, for $\bar{f} \in F_3$

$$\bar{f}(\tau_2) = \sum_{l=1}^{m} g_l(z) = \sum_{\substack{l=1 \\ l \text{ even}}}^{n} g_l(z) = (n/2) f(n) = cn/2.$$

So clearly $\|\tau_2\| \geqq cn/2$.

§6. Proof of the Main Theorem

PROOF OF THEOREM 1.7.

Stage I. As $\diamondsuit_{\aleph_1}$ holds, there is a sequence $\langle I^\delta : \delta < \omega_1 \text{ limit} \rangle$ such that:

(α) I^δ is a set of m-sequences from δ,

(β) for every m and set I of m-sequences from ω_1, $\{\delta : I \restriction \delta = I^\delta\}$ is a stationary subset of ω_1.

Stage II. The induction conditions. We define by induction on $\alpha, M_\alpha, M_\alpha^n$ ($n < \omega$) and Γ_α such that the following conditions are satisfied:

(A) (1) $M_\alpha \in K^1(\aleph_0)$ has universe $\subseteq \omega(1+\alpha)$, M_α increasing and continuous, $|M_\alpha| \cap \omega(1+\beta) = |M_\beta|$ for $\beta < \alpha$,

(2) $S_\alpha \subseteq \omega$ infinite, $M_\alpha = \bigcup_{n \in S_\alpha} M_\alpha^n$; for $n \in S_\alpha$: M_α^n increase with n; $M_\alpha^n \in K$,

(3) $\alpha = \bigcup_{n \in S_\alpha} w_\alpha^n$, w_α^n is finite, $\neq \emptyset$, increases with n, for $\beta \in w_\alpha^n$ ($n \in S_\beta$ and) $w_\beta^n = w_\alpha^n \cap \beta$ ($\forall \gamma + 1 \in w_\alpha^n$) ($\gamma \in w_\alpha^n$) and $0 \in w_\alpha^n$; S_α infinite and for $\beta < \alpha$, $S_\beta - S_\alpha$ is finite,

(4) for $n \in S_\alpha$, $\beta \in w_\alpha^n : M_\beta^n \subseteq M_\alpha^n$,

(5) for α limit, $M_\alpha^n = \bigcup_{\beta \in w_\alpha^n} M_\beta^n = M_\gamma^n$ where $\gamma = \operatorname{Max} w_\alpha^n$,

(6) if $\alpha = \beta + 1$, $\beta \in w_n^\alpha$ then $R(M_\beta^n, M_\alpha^n, b_\beta)$ (so $b_\beta \in M_\alpha^n$);

(B) (1) if $\alpha = \beta + 1$, $M_\alpha^n \subseteq N \in K$, and N cannot be embedded into M_α over M_α^n, then for some $k > n$ ($k \in S_\alpha$), N cannot be embedded over M_α^n into any $N' \in K$ extending M_α^k satisfying $R(M_\beta^k, N', a_\beta)$,

(2) if $\alpha = 0$, $M_\alpha^n \subseteq N \in K$, and N cannot be embedded into M_α over M_α^n, then for some $k > n$, N cannot be embedded over M_α^n into any extension $N' \in K$ of M_α^k,

(3) if $\alpha = 0$, $N \in K$, and N cannot be embedded into M_α, then for some $k > n$, N cannot be embedded into any extension $N' \in K$ of M_α^n,

(4) for every α and triple $(N_0, N_1, a) \in R$, $N_0 \subseteq M_\alpha$, $N_0 \in K$, $N_1 \in K$, for arbitrarily large $\beta > \alpha$, there is an isomorphism f from N_1 over N_0 into $M^0_{\beta+1}$, $f(a) = b_\beta$.

Before we continue to list the conditions, we define, for $\delta \leqq \alpha < \omega_1$ (when M^n_β ($\beta \leqq \alpha$, $n < \omega$) have already been defined):

$W^\alpha_\delta = \{(\bar{N}, p, h) : \bar{N}$ an M_δ-candidate, $p \leqq k(\bar{N})$, $h \restriction N_0 =$ the identity, h an embedding of N_p into M_α ; for some n, for each $l < p$ for some β_l, $\delta \leqq \beta_l < \alpha$, $h(a_l) = b_{\beta_l}$, $h(\bar{N}_{l+1}) = M^n_{\beta_l+1}$ and $\{\beta_l + 1 : l < p\} = w^n_\alpha - \delta - \{\gamma$: for some limit $\beta \in w^n_\alpha$, $\gamma = \mathrm{Max}(w^n_\alpha \cap \beta)\}\}$;

$n(h)$ will be the n mentioned above.

REMARK. Note that necessarily $\beta_l < \beta_{l+1}$.

We say $(\bar{N}, \bar{b})$ is an M-witness if $\bar{N}$ is an M-candidate, $\bar{b} \in \bar{N}$; and

(C) (1) Γ_α is a countable family, increasing with α,

(2) each member of Γ_α has the form (D, δ), δ a limit ordinal $\leqq \alpha$ or is zero, and D a set of M_δ-witnesses, $(\forall \bar{N}')[(\bar{N}, \bar{b}) \in D$, $\bar{N} \leqq \bar{N}'$, $N'_0 \subseteq M \Rightarrow (\bar{N}', \bar{b}) \in D]$ and $\bar{b} \in N_0$, $\bar{N}$ an M_δ-witness implies $(\bar{N}, \bar{b}) \in D$,

(3) M_α satisfies each $(D, \delta) \in \Gamma_\alpha$, which means: for every $(\bar{N}, \bar{b})$ an M_α-witness and p, h and $r < \omega$ s.t. $(\bar{N}, p, h) \in W^\alpha_\delta$, there is $\bar{N}' \geqq N$ and h' extending h s.t. $(\bar{N}', p, h') \in W^\alpha_\delta$ and $(\bar{N}', \bar{b}) \in D$, $n(h') \geqq r$,

(4) if δ is limit and M_α satisfies (D_δ, δ) (see 3) then $(D_\delta, \delta) \in \Gamma_\delta$, where $D_\delta = \{(\bar{N}, \bar{b}) : (\bar{N}, \bar{b})$ is an M_δ-witness and: $\bar{b} \in N_0$ or there is no embedding f of $\bar{N}$ into M_δ over N_0 with $f(\bar{b}) \in I_\delta\}$,

(5) $\Gamma_0 = \{(D_0, 0)\}$, $D_0 = \{(\bar{N}, b) : \bar{N}$ a candidate, $b \in N_{k(\bar{N})-1}$ or for every $\bar{N}' \geqq \bar{N}$, $b \notin N'_{k(\bar{N})-1}\}$.

Stage III. Why carrying II suffices. Suppose we have carried the induction. We let $M^* = \bigcup_{\alpha<\omega_1} M_\alpha$ and b_α, M^n_α has been defined. Let us check the conditions one-by-one:

(a) We know that $M_\alpha \in K^1(\aleph_0)$ (by (A)(1)), M_α increasing (by (A)(1)). By (A)(2), (A)(6), $b_\alpha \in M_{\alpha+1} - M_\alpha$ hence $\|M^*\| = \aleph_1$. By the definition of $K^1(\lambda)$ (see 1.3) we finish.

(c) Immediate by (A)(1) and (A)(6).

(d) By (B)(4).

(e) By (B)(1), (B)(3) (for α limit it is automatic as M_α is increasing continuous).

(f) By (B)(2), (B)(3).

So we are left with proving (b). Suppose I, m form a counterexample. So I is a set of m-sequences from ω_1, $I \subseteq M^*$ and there is no witness for (M^*, I). Hence there is a function g whose domain is the set of M^*-witnesses $(\bar{N}, \bar{b})$ $(l(\bar{b}) = m)$, $\bar{N} \leqq \bar{N}'$ where $\bar{N}' = g(\bar{N}, \bar{b})$, $N'_0 \subseteq M^*$, $[\bar{b} \not\in N'_0 \Rightarrow (\forall \bar{N}'' \geqq \bar{N}')\bar{b} \not\in N''_0]$ and if $\bar{b} \not\in N'_0$ there is no embedding f of $\bar{N}'$ into M^* over N'_0 such that $f(\bar{b}) \in I$. Clearly every $\bar{N}''$ enjoys this property if $N''_0 \subseteq M^*$.

Let $C = \{\alpha < \omega_1 : $(a) α a limit ordinal, $\omega(1 + \alpha) = \alpha$, (b) for every M_α-witness $(\bar{N}, \bar{b})$, $N'_0 \subseteq M_\alpha$ where $\bar{N}' = g(\bar{N}, \bar{b})$, (c) for every $\beta < \alpha < \gamma$ and $n < k$, $\bar{c}_1, \ldots, \bar{c}_n \in M^n_\gamma$ there is γ', $\beta < \gamma' < \alpha$, and isomorphism h of M^n_γ onto $M^n_{\gamma'}$ s.t. $\bar{c}_l \in I \Leftrightarrow h(\bar{c}_l) \in I$ and if $w^n_\gamma = \{\varepsilon_1, \ldots, \varepsilon_h\}$ then $w^n_{\gamma'} = \{\varepsilon'_1, \ldots, \varepsilon'_h\}$, $\varepsilon_l < \beta \vee \varepsilon'_l < \beta \Rightarrow \varepsilon_l = \varepsilon'_l$, h maps $M^n_{\varepsilon_l}$ onto $M^n_{\varepsilon'_l}$, b_{ε_l} to $b_{\varepsilon'_l}$, and $\varepsilon_l < \beta \Rightarrow h \restriction M^n_{\varepsilon_l} =$ the identity$\}$. Clearly C is a closed unbounded subset of ω_1 (remembering there are essentially $\aleph_0$ M_α-candidates, as K has only countably many members up to isomorphism and each $N_0 \in K$ is f.g.).

So for some $\delta \in C$, $I^\delta = \{\bar{b} : \bar{b} \in I, \bar{b} \in M_\delta\}$. Now using $g \restriction \{(N, \bar{b}) : (\bar{N}, \bar{b})$ and M_δ-witness$\}$ we see that M_δ satisfies (D_δ, δ) (see definition in (C)(3) where D_δ is defined as in (C)(4)). Hence by (C)(4) $(D_\delta, \delta) \in \Gamma_\delta$. As I is uncountable, there is $\bar{c} \in I$, $\bar{c} \not\in M_\delta$.

So for some $\alpha > \delta$, $\bar{c} \in M_\alpha$, w.l.o.g. α is minimal and for some n, $\bar{c} \in M^n_\alpha$. W.l.o.g. $\delta + 1, \delta \in w^n_\alpha$, and let $w^n_\alpha - \delta - \{\beta : \beta$ a limit ordinal$\} = \{\beta_1 + 1, \beta_2 + 1, \ldots, \beta_k + 1\}$. Note that if $\gamma \in w^n_\alpha$ is a limit ordinal then $M^n_\gamma = M^n_\beta$ where $\beta = \mathrm{Max}(w^n_\alpha \cap \gamma)$. Of course $\beta_1 < \beta_2 < \cdots$ hence $\beta_1 = \delta$. So

$$\bar{N} \stackrel{\text{def}}{=} \langle M^n_\delta, M^n_{\beta_1+1}, \ldots, M^n_{\beta_k+1}, a_{\beta_1}, \ldots, a_{\beta_k} \rangle$$

is a candidate over M_δ, and $\bar{c} \in \bar{N}$. So with h the identity over $M^n_{\beta_k+1}$, $p = k + 1$, $(\bar{N}, p, h) \in W^\alpha_\delta$. So $\bar{c} \in M_{\beta_k+1}$, and $\beta_k < \alpha$ so $\beta_k + 1 = \alpha$ hence by (C)(3) apply to $(D_0, 0)$ (which belongs to $\Gamma_0 \subseteq \Gamma_\alpha$); we could have chosen n large enough so that for no $\bar{N}' \geqq \bar{N}$, $\bar{b} \in N'_0$. So we can apply (C)(3): there is $\bar{N}' \geqq \bar{N}$ and h' extending h so that $(\bar{N}', p, h') \in W^\alpha_\delta$ and $(N', \bar{c}) \in D_\delta$. So $\bar{N} \leqq h'(\bar{N}')$ (as h is the identity), $(\bar{N}', \bar{b}) \in D_\delta$, and between the definition of D_δ, choice of δ and C we get a contradiction.

Stage IV. Suppose everything is defined for $\beta \leqq \alpha$ and we define, for $\alpha + 1$, $\Gamma_{\alpha+1} = \Gamma_\alpha$: there is no problem to define $w^n_{\alpha+1}$ $(n < \omega)$ as required in (A)(3). Let, for $n \in S_\alpha$,

$$w^n_\alpha - \{\delta : \delta \text{ limit or } 0\} = \{\varepsilon(\alpha,n,1)+1,\dots,\varepsilon(\alpha,n,k_\alpha,n)+1\} \quad \text{(increasing)},$$

$$\bar N^m = \langle M^n_0, M^n_{\varepsilon(\alpha,n,1)+1},\dots, M^n_{\varepsilon(\alpha,n,k_\alpha(n))+1}, a_{\varepsilon(\alpha,n,1)},\dots, a_{\varepsilon(\alpha,n,k_\alpha(n))}\rangle.$$

We define by induction on n, $k_n \in S_\alpha$, $M^{k_n}_{\alpha,*} \in K$, $k_n < k_{n+1}$, e^n_α an embedding of $M^{k_n}_{\alpha,*}$ into $M^{k_{n+1}}_{\alpha,*}$ over $M^{k_n}_\alpha$ s.t. $(M^{k_n}_\alpha, M^{k_n}_{\alpha,*}, a_\alpha) \in R$.

Later $M_{\alpha+1}$ is the direct limit of $\langle M^{k_n}_{\alpha,*} : n < \omega\rangle$ by the $e^n_{\alpha+1}$'s, $M^{k_n}_{\alpha+1}$ the images of $M^{k_n}_{\alpha,*}$. By standard bookkeeping in each stage we get a specific requirement (each appearing infinitely (often) from some condition each contributing $\aleph_0$ specific requirements) and we have to prove that we can fulfill it.

In choosing M^0_α we take care of (B)(4).

(A)(1) will follow from (A)(4),

(A)(2) by the requirements on the induction on n,

(A)(3) is already fulfilled,

(A)(4) by the requirements on the induction on n,

(A)(5) irrelevant,

(A)(6) by the requirements on the induction on n,

(B)(1)–(4) easy,

(C)(1) we have satisfied ($\Gamma_{\alpha+1} = \Gamma_\alpha$),

(C)(2) similar to (C)(1).

(C)(3) Suppose $(D,\delta) \in \Gamma_\alpha$, and $(\bar N, p, h) \in W^{\alpha+1}_\delta$, $(\bar N, \bar b)$ an M_δ-witness. Note that for each $n < \omega$ we can list the $(\bar N, p, h)$ for which h embeds $\bar N$ into $M^{k_n}_{\alpha+1}$ so no vicious circle arises. So we are given n, $(N, \bar b)$, $(\bar N, p, h)$ s.t. h embeds $\bar N$ into $M^{k_n}_{\alpha+1}$, $(\bar N, \bar b)$ an M_δ-witness.† W.l.o.g. $h(a_{p-1}) = b_\alpha$ (otherwise use (C)(3) for α!).

Let $w^{k_n}_{\alpha+1} - \{\gamma : \gamma \text{ limit}\} = \{\alpha_1,\dots,\alpha_p\}$ (increasing) so $\alpha_p = \alpha+1$.

Choose k large enough, $k \in S_\alpha$. We define by induction on $l \leqq k(\bar N)$, N'_l, h'_l for $l < p$ such that

$$\langle N'_l,\dots,N'_0, a_{l-1},\dots,a_0\rangle \geqq \langle N_l,\dots,N_0,a_{l-1},\dots,a_0\rangle,$$

h'_l extends $h \restriction N_l$, h'_l maps N'_l onto $M^k_{\alpha_l+1}$ where $h(a_l) = h_{\alpha_l}$. By (A)(6) this is impossible.

Now we define M', e', where e' embeds $M^{k_n}_{\alpha,*}$ into M' over $M^{k_n}_\alpha, M^k_\alpha \subseteq M'$, $R(M^k_\alpha, M', b_\alpha)$ (see Definition 1.2(1)(c)). Then we can define N'_p and $h'_p \supseteq h'_{p-1} \cup e'$, h'_p an isomorphism from N_p onto M'. Similarly (by Definition 1.2(1)(c)) we can define $\bar N', \bar N \leqq \bar N'$ (N'_l ($l \leqq p$) are already defined).

†And for some δ, $(\bar N, p, b) \in W^{\alpha+1}_\delta$; so $h \restriction N_0 =$ the identity and for some m

$$l < p \Rightarrow h(N_l) \in \{M^m_\gamma : \gamma \in w^m_{\alpha+1}\},$$

$$h(a_l) \in \{b_\gamma : \gamma \in w^m_{\alpha+1}\}.$$

Now apply (C)(3) for α, h'_{p-1}, $\langle N_{p-1},\dots,N_0,a_{p-2},\dots,a_0\rangle$ and get $\bar{N}'',h'',n(h'')\geqq r$; $n(h'')\in S_\alpha$.

Let $k_{n+1}=n(h'')$. Extending h'' will give us $M^{k_{n+1}}_{\alpha,*}$ and we have no problem.

(C)(4) Trivial.

(C)(5) Trivial.

Stage V. α limit. Easy.

Stage VI. $\alpha=0$. Easy.

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